

Strongly J-clean matrices over 2-projective-free rings

Marjan Sheibani, Huanyin Chen and Rahman Bahmani

Abstract

An element a of a ring is strongly J-clean if it is the sum of an idempotent and an element in the Jacobson radical that commutes. We characterize the strongly J-clean 2×2 matrices over noncommutative 2-projective-free rings. For a 2-projective-free ring R , $A \in M_2(R)$ is strongly J-clean if and only if $A \in J(M_2(R))$, or $I_2 - A \in J(M_2(R))$, or A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R)$, $\mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. Strongly J-clean 2×2 matrices over power series are therefrom investigated. We prove that if R is a 2-projective-free wb-ring then $A(x) \in M_2(R[[x]])$ is strongly J-clean if and only if $A(0) \in M_2(R)$ is strongly J-clean.

Keywords: strongly J-clean matrix; 2-projective-free ring; quadratic equation, power series.

2010 Mathematics Subject Classification: 15E50, 16U60.

1 Introduction

An element $a \in R$ is called *strongly J-clean* (*clean*) if there exists an idempotent $e \in R$ such that $a - e \in J(R)$ ($U(R)$) and $ae = ea$. A ring R is *strongly J-clean* (*clean*) provided that every element in R is strongly J-clean (clean). A ring R is uniquely clean if for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$, which was introduced by Anderson and Camillo [1]. Nicholson and Zhou proved that a ring R is uniquely clean if and only if for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in J(R)$ [10, Theorem 20]. Evidently, $\{\text{uniquely clean rings}\} \subsetneq \{\text{strongly J-clean rings}\} \subsetneq \{\text{strongly clean rings}\}$. The inclusions are both proper. For instances, The ring $T_2(\mathbb{Z}_2)$ of all 2×2 up triangular matrices over \mathbb{Z}_2 is strongly J-clean, while it is not uniquely clean [4, Corollary 16.4.24]; and that \mathbb{Z}_3 is strongly clean, while it is not strongly J-clean. Thus, the class of strongly J-clean rings is a medium between those of uniquely clean rings and strongly clean rings. On the other hand, for an arbitrary ring R ,

one easily checks that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(R)$ is not strongly J-clean. Hence, it is attractive to study when a matrix over a ring can be written in these special forms. In fact, Strong J-cleanness (cleanness) of 2×2 matrices over commutative local rings are studied by many authors. [9] and [13] investigated strongly clean decompositions of 2×2 matrices over local rings. Recently, strong cleanness of matrices over a general ring was discussed in [7] and [8]. Furthermore, strongly J-clean matrices over noncommutative local rings were studied in [5].

Let R be a ring with an identity. We say that a ring R is a *2-projective-free ring* if every 2-generated projective R -module is free of constant rank. A ring R is *projective-free* if every finitely generated projective R -module is free of constant rank. Thus, projective-free rings form a subclass of 2-projective-free rings. Then we see that division rings (including fields), local rings, Bézout domain (i.e., domain in which every finitely generated right ideal is principal, including principal ideal domains, valuation rings, the ring of entire functions and the ring $\overline{\mathbb{Z}}$ of all algebraic integers), polynomial rings of a principal ideal domains are all 2-projective-free. The motivation of this article is to explore strongly J-clean decompositions of 2×2 matrices over such new kind of rings.

In Section 2, we shall investigate elementary properties of 2-projective-free rings which will be used in the sequel. In particular, we prove that every PSF ring is 2-projective-free. This provides a large class of such rings.

We extend, in Section 3, the main results of strongly J-clean matrices over local rings in [5] to noncommutative 2-projective-free rings. For a 2-projective-free ring R , we show that $A \in M_2(R)$ is strongly J-clean if and only if $A \in J(M_2(R))$, or $I_2 - A \in J(M_2(R))$, or A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R)$, $\mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. Strongly J-clean matrices over a commutative 2-projective-free ring are thereby characterized.

Section 4 is concern on strongly J-clean matrices over power series of 2-projective-free rings. Let R be a ring, and let $f(x) \in R[[x]]$. If $f(0) \in R$ is optimally J-clean, we prove that $f(x) \in R[[x]]$ is strongly J-clean. Let R be a ring, and let $\alpha, \beta \in R$. $\ell_\alpha - r_\beta : R \rightarrow R$ is the group homomorphism given by $x \mapsto \alpha x - x\beta$ for any $x \in R$. We say that R is a *weakly bleached ring* (*wb-ring*, for short) provided that for any $\alpha \in J(R)$, $\beta \in 1 + J(R)$, $\ell_\alpha - r_\beta$ and $\ell_\beta - r_\alpha$ are surjective. For instance, every commutative ring is a wb-ring. This concept was firstly introduced only for general local rings (not necessary be commutative) [13]. Let R be a 2-projective-free wb-ring. We shall prove that $A(x) \in M_2(R[[x]])$ is strongly J-clean if and only if so is $A(0) \in M_2(R)$ is strongly J-clean.

Throughout, all rings are associative with an identity. $M_n(R)$ will denote the ring of all $n \times n$ full matrices over R with an identity I_n . $GL_n(R)$ stands for the n -dimensional

general linear group of R . Let M be a right module. $\text{end}(M)$ and $\text{aut}(M)$ stand for the ring of endomorphism and automorphism of M , respectively. We always use $[a, b]$ to stand for the commutator $ab - ba$ for any $a, b \in R$.

2 2-Projective-free rings

The purpose of this section is to investigate elementary properties of 2-projective-free rings which will be used in the sequel. A ring R is called *2-IBN* if $R^m \cong R^n$ ($m, n = 1, 2$) implies that $m = n$. We begin with

Proposition 2.1 *A ring is 2-projective-free if and only if R is 2-IBN and every idempotent 2×2 matrix over R admits a diagonal reduction to I_2 or $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.*

Proof This is similar to that of projective-free rings [6, Proposition 2.6]. □

Corollary 2.2 *Every 2-projective-free ring R has only trivial idempotents.*

Proof Let $0, 1 \neq e \in R$ be an idempotent. Then eR is a trivially 2-generated projective module. Hence, eR is free. Thus, $eR = 0$ or $eR \cong R$. Likewise, $(1 - e)R = 0$ or $(1 - e)R \cong R$. This implies that $eR \cong (1 - e)R \cong R$, and so $R \cong eR \oplus (1 - e)R \cong R^2$. By Proposition 2.1, R is 2-IBN. This gives a contradiction, hence the result. □

As an immediate consequence, we deduce that every 2-projective-free ring is directly finite, i.e., every right or left invertible element is invertible.

Lemma 2.3 *Let $I \subseteq J(R)$. If R/I is 2-projective-free, then so is R .*

Proof Let P be a 2-generated projective R -module then P/IP is a 2-generated projective R/I -module. As R/I is 2-projective-free, we get $P/IP \cong (R/I)^m \cong R^m/IR^m$. Since $I \subseteq J(R)$, we deduce that $P \cong R^m$, as desired. □

Theorem 2.4 *Let R be 2-projective-free. Then $R[[x]]$ is 2-projective-free.*

Proof Let $\varphi : R[[X]] \rightarrow R$ be a ring homomorphism that is defined by $\varphi(f) = f(0)$, then φ is surjective. Since invertible elements of $R[[X]]$ are those whose constant terms are invertible in R , we have $\ker(\varphi) \subseteq J(R[[x]])$. Clearly, $R[[X]]/\ker(\varphi) \cong R$. It follows by Lemma 2.3 that $R[[X]]$ is 2-projective-free, as asserted. □

An R -module P is *stably free* if there exists natural numbers m, n such that $P \oplus R^m \cong R^n$. It is well known that the ring $\mathbb{Z}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ has stable free modules which are not free [11, Exercise 1.4.23]. A commutative ring R is called a *PSF ring* if each finitely generated projective module is stably free, i.e., the *Grothendieck group* $K_0(R) = \mathbb{Z}[R]$. Every projective-free ring is PSF. In fact, $\{\text{PSF rings}\} \cap \{\text{Hermite rings}\} = \{\text{projective-free rings}\}$. The following result provides a large class of 2-projective-free rings.

Theorem 2.5 *Every PSF ring is 2-projective-free.*

Proof Let P be a 2-generated projective module over a PSF ring R . Then there exists an epimorphism $\varphi : R^2 \rightarrow P$, and so $P \oplus Q \cong R^2$, where $Q = \ker(\varphi)$. By hypothesis, P is stably free. Write $P \oplus R^m \cong R^n$. Then P is of constant rank, i.e., $\text{rank}(P) = n - m$. Likewise, Q is of constant rank. Clearly, $\text{rank}(P) + \text{rank}(Q) = 2$. If $\text{rank}(P) = 0$, then $P = 0$. If $\text{rank}(P) = 2$, then $\text{rank}(Q) = 0$, and so $Q = 0$. It follows that $P \cong R^2$. If $\text{rank}(P) = 1$, then $P \oplus R^m \cong R^{m+1}$, and so P is free, as every stably free module of rank 1 over a commutative ring is free, and we are through. \square

Example 2.6 Let $R = \mathbb{Z}[\sqrt{-3}]$. It is well known that $K_0(R) \cong \mathbb{Z}$, $[R] \mapsto 1$, and so $K_0(R) = \mathbb{Z}[R]$. Thus, R is PSF. In terms of Theorem 2.5, R is 2-projective-free. In this case, R is not principal domain, even it is not a Dedekind domain [11, Exercise 1.4.21].

3 Strongly J-clean matrices

The aim of this section is to characterize a single strongly J-clean matrix over 2-projective-free rings in terms of the solvability of quadratic equation.

Theorem 3.1 *Let R be 2-projective-free. Then $A \in M_2(R)$ is strongly J-clean if and only if $A \in J(M_2(R))$ or $I_2 - A \in J(M_2(R))$ or A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in 1 + J(R), \beta \in J(R)$.*

Proof \Leftarrow If $A \in J(M_2(R))$, then $A = 0 + A$ is strongly J-clean. If $I_2 - A \in J(M_2(R))$, then $A = I_2 + (A - I_2)$ is strongly J-clean. If A is similar to a matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $\alpha \in 1 + J(R), \beta \in J(R)$, then there exists some $U \in GL_2(R)$ such that

$$A = U^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U + U^{-1} \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta \end{pmatrix} U \text{ is strongly J-clean.}$$

\implies By hypothesis, there exists an idempotent $E \in M_2(R)$ and a $W \in J(M_2(R))$ such that $A = E + W$ with $EW = WE$. Suppose that A and $I_2 - A$ are not in $J(M_2(R))$. Since R is 2-projective-free, by virtue of Proposition 2.1, there exists $U \in GL_2(R)$ such that $UEU^{-1} = \text{diag}(1, 0)$. Hence, $UAU^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + UWU^{-1}$. Set $V = (v_{ij}) := UWU^{-1}$. Then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; whence, $v_{12} = v_{21} = 0$ and $v_{11}, v_{22} \in J(R)$. Therefore A is similar to $\begin{pmatrix} 1 + v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$, which completes the proof. \square

Let R be any ring with $J(R) = 0$ (i.e., division ring). Then $A \in M_2(R)$ is strongly J-clean if and only if A is an idempotent matrix. Further, we derive

Corollary 3.2 *Let R be any 2-projective-free commutative ring with $J(R) = 0$ (e.g., \mathbb{Z}), and let $A \in M_2(R)$. Then A is strongly J-clean if and only if $A = 0$ or I_2 , or $\begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix}$ with $bc = a - a^2$.*

Proof \implies In view of Theorem 3.1, A is strongly J-clean if and only if $A = 0$ or I_2 , or A is similar to $\text{diag}(1, 0)$. If $UAU^{-1} = \text{diag}(1, 0)$ for a $U \in GL_2(R)$, then $\det(A) = 0$ and $\text{tr}(A) = 1$. Thus,

$$A = \begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix} \text{ with } bc = a - a^2.$$

\Leftarrow One easily checks that A is an idempotent matrix, and then strongly J-clean. \square

Lemma 3.3 [5, Theorem 2.1] *Let $E = \text{end}({}_R M)$, and let $\alpha \in E$. Then the following are equivalent:*

- (1) α is strongly J-clean in E .
- (2) $M = P \oplus Q$ where P and Q are α -invariant, and $\alpha|_P \in J(\text{end}(P))$ and $(1_M - \alpha)|_Q \in J(\text{end}(Q))$.

Lemma 3.4 *Let R be 2-projective-free, and let $A \in M_2(R)$ be strongly J-clean. Then $A \in J(M_2(R))$ or $I_2 - A \in J(M_2(R))$ or A is similar to a matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in 1 + J(R)$.*

Proof Suppose that $A, I_2 - A \notin J(M_2(R))$. By virtue of Theorem 3.1, we have a $P \in GL_2(R)$ such that $PAP^{-1} = \begin{pmatrix} 1+\alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha, \beta \in J(R)$. Thus, we check that

$$UAU^{-1} = \begin{pmatrix} 0 & -(1+\alpha)(1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta) \\ 1 & (1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta) + (1+\alpha) \end{pmatrix},$$

where

$$U = \begin{pmatrix} 1 & -1-\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1-\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1+\alpha-\beta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Set $\lambda = -(1+\alpha)(1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta)$ and $\mu = (1+\alpha-\beta)^{-1}\beta(1+\alpha-\beta) + (1+\alpha)$. Then $\lambda \in J(R)$ and $\mu \in 1 + J(R)$, as desired. \square

We come now to the main result of this section.

Theorem 3.5 *Let R be 2-projective-free. Then $A \in M_2(R)$ is strongly J-clean if and only if*

- (1) $A \in J(M_2(R))$, or
- (2) $I_2 - A \in J(M_2(R))$, or
- (3) A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Proof Suppose that $A \in M_2(R)$ is strongly J-clean, and that $A, I_2 - A \notin J(M_2(R))$. It follows by Lemma 3.4 that A is similar to the matrix $B = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$. Hence, $B \in M_2(R)$ is strongly J-clean. In view of Lemma 3.3, we have $2R = C \oplus D$ where $(I_2 - B)|_C \in J(\text{end}(C))$ and $B|_D \in J(\text{end}(D))$. Thus, $B|_C \in \text{aut}(C)$ and $(I_2 - B)|_D \in \text{aut}(D)$. Since R is 2-projective-free, it follows by Proposition 2.1 that C and D are free. As $B, I_2 - B \notin J(M_2(R))$, we see that $C, D \cong R$. Assume that (a, b) and (c, d) are bases of C and D , respectively. Then $C = R(a, b), D = R(c, d)$. Then

$$R(a, b) \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = R(a, b).$$

Set $\overline{R} = R/J(R)$. Then

$$\overline{R}(\overline{a}, \overline{b}) \subseteq \overline{R}(\overline{1}, \overline{1}).$$

Similarly,

$$\overline{R}(\overline{c}, \overline{d}) \subseteq \overline{R}(\overline{1}, \overline{0}).$$

Write $(\overline{a}, \overline{b}) = s(\overline{1}, \overline{1})$ and $(\overline{c}, \overline{d}) = t(\overline{1}, \overline{0})$. Then

$$(\overline{1}, \overline{1}) = z(\overline{a}, \overline{b}) + z'(\overline{c}, \overline{d}) = zs(\overline{1}, \overline{1}) + z't(\overline{1}, \overline{0}).$$

This implies that $1 - zs \in J(R)$, and so $s \in R$ is left invertible. Hence, $s \in U(R)$, as R is directly finite. Clearly, $a - s, b - s \in J(R)$, and so $1 - a^{-1}b \in J(R)$. $C = R(a, b) = R(1, \alpha)$, where $\alpha = a^{-1}b \in 1 + J(R)$. Analogously, $D = R(1, \beta)$, where $\beta = c^{-1}d \in J(R)$. As C is B -invariant, we see that

$$(1, \alpha) \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = r(1, \alpha)$$

for some $r \in R$. It follows that $\alpha = r$ and $\lambda + \alpha\mu = r\alpha$, and therefore $\alpha^2 - \alpha\mu - \lambda = 0$, i.e., $x^2 - x\mu - \lambda = 0$ has a root $\alpha \in 1 + J(R)$. Likewise, this equation has a root $\beta \in J(R)$, as desired.

Conversely, if (1) or (2) holds then $A \in M_2(R)$ is strongly J-clean, and so we assume (3) holds. As strong J-cleanness is invariant under similarity, we will suffice to check if $B = \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is strongly J-clean. By hypothesis, the equation $x^2 - x\mu - \lambda = 0$ has roots $c \in J(R)$ and $d \in 1 + J(R)$. Then $c^2 - c\mu - \lambda = 0$ and $d^2 - d\mu - \lambda = 0$. Choose $C = R(1, c)$ and $D = R(1, d)$. Since

$$(1, c) \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = c(1, c) \in C,$$

C is B -invariant. Similarly, D is B -invariant. If $r(1, c) = s(1, d) \in C \cap D$, then $r = s$ and $rc = sd$; hence, $r(c - d) = 0$. Since $c - d \in U(R)$, we get $r = 0$. Thus, $C \cap D = 0$. Let $(a, b) \in 2R$. Choose $s = (b - ac)(d - c)^{-1}$ and $r = a - s$. Then $(a, b) = r(1, c) + s(1, d) \in C \oplus D$. Hence, $2R = C \oplus D$. Let $\gamma \in \text{end}(C)$. Then

$$\begin{aligned} 1_C - B|_C \gamma : C &\rightarrow C; \\ r(1, c) &\mapsto r(1, c) - rc(1, c)\gamma. \end{aligned}$$

Write $(1, c)\gamma = b(1, c)$ for a $b \in R$. If $(r(1, c))(1_C - B|_C \gamma) = 0$, then $r(1, c) - rc b(1, c) = 0$; hence, $r(1 - cb)(1, c) = 0$. It follows from $c \in J(R)$ that $r = 0$, and so $r(1, c) = 0$. Thus, $1_C - B|_C \gamma$ is monomorphic. For any $r(1, c) \in C$, we see that

$$(r(1 - cb)^{-1}(1, c))(r(1, c))(1_C - B|_C \gamma) = r(1, c).$$

This implies that $1_C - B|_C \gamma$ is epimorphic. As a result, $1_C - B|_C \gamma$ is isomorphic. We infer that $B|_C \in J(\text{end}(C))$. Similarly, $(I_2 - B)|_D \in J(\text{end}(D))$. In light of Lemma 3.4, $B \in M_2(R)$ is strongly J-clean, as needed. \square

A matrix $A \in M_2(R)$ is *cyclic* if there exists a column α such that $(\alpha, A\alpha) \in GL_2(R)$. For instance, $\begin{pmatrix} * & * \\ u & * \end{pmatrix} \in M_2(R)$ is cyclic for any $u \in U(R)$.

Corollary 3.6 *Let R be a 2-projective-free ring, and let $A \in M_2(R)$. If R is commutative, then A is strongly J-clean if and only if*

- (1) $A \in J(M_2(R))$, or
- (2) $I_2 - A \in J(M_2(R))$, or
- (3) A is cyclic and $x^2 - \text{tr}(A)x + \det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Proof Suppose that A is strongly J-clean. If $A, I_2 - A \notin J(M_2(R))$, then A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$, by Theorem 3.5. In view of [4, Lemma 7.4.6], A is cyclic. As R is commutative, we see that $\text{tr}(A) = \mu$ and $\det(A) = -\lambda$, and so $x^2 - \text{tr}(A)x + \det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Conversely, if $A \in J(M_2(R))$ or $I_2 - A \in J(M_2(R))$ then A is strongly J-clean. We now assume that A is cyclic and $x^2 - \text{tr}(A)x + \det(A) = 0$ has a root α in $J(R)$ and a root β in $1 + J(R)$. In view of [4, Lemma 7.4.6], A is isomorphic to a companion matrix $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$. This shows that $\mu = \text{tr}(A)$ and $\det(A) = -\lambda$. Since

$$\alpha^2 - \text{tr}(A)\alpha + \det(A) = 0 \text{ and } \beta^2 - \text{tr}(A)\beta + \det(A) = 0,$$

we get $\text{tr}(A) = \alpha + \beta$ and $\det(A) = \alpha\beta$. Hence, $\mu = \alpha + \beta \in 1 + J(R)$ and $\lambda = -\alpha\beta \in J(R)$. Therefore we complete the proof, by Theorem 3.5. \square

Let R be a PSF ring, and let $A \in M_2(R)$. It follows from Corollary 3.6 that A is strongly J-clean if and only if $A \in J(M_2(R))$, or $I_2 - A \in J(M_2(R))$, or A is cyclic and $x^2 - \text{tr}(A)x + \det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. The following is a dual of [5, Theorem 2.5].

Corollary 3.7 *Let R be a local ring. Then $A \in M_2(R)$ is strongly J-clean if and only if*

- (1) $A \in J(M_2(R))$, or
- (2) $I_2 - A \in J(M_2(R))$, or
- (3) A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - \mu x - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Proof Suppose that $A \in M_2(R)$ is strongly J-clean. Then there exist an idempotent $E \in M_2(R)$ and $W \in M_2(J(R))$ such that $A = E + W$ and $AE = EA$. Hence, $(A^o)^T = (E^o)^T + (W^o)^T$. One easily checks that $(E^o)^T \in M_2(R^{op})$ is an idempotent matrix and $(W^o)^T \in M_2(J(R^{op}))$. Furthermore, $(A^o)^T(E^o)^T = (E^o)^T(A^o)^T$. Therefore, $(A^o)^T \in M_2(R^{op})$ is strongly J-clean. Clearly, R^{op} is local, and then it is 2-projective-free. Applying Theorem 3.6 to $(A^o)^T \in M_2(R^{op})$. Then $(A^o)^T \in J(M_2(R^{op}))$, or $I_2^o - (A^o)^T \in J(M_2(R))$, or $(A^o)^T$ is similar to $\begin{pmatrix} 0^o & \lambda^o \\ 1^o & \mu^o \end{pmatrix}$ where $\lambda^o \in J(R^{op}), \mu \in 1^o + J(R^{op})$, and the equation $x^2 - x\mu^o - \lambda^o = 0$ has a root in $J(R^{op})$ and a root in $1^o + J(R^{op})$. Thus, $A \in J(M_2(R))$, or $I_2 - A \in J(M_2(R))$, or A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in 1 + J(R)$, and the equation $x^2 - \mu x - \lambda = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

The converse is proved by a similar route. \square

4 Power Series Rings

This section is concern on strongly J-clean matrices over power series rings. We say that an element $a \in R$ is *optimally J-clean* provided that there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in R$ such that $[a, c] = [e, b]$. A ring R is *optimally J-clean* provided that every element in R is optimally J-clean. Every uniquely clean ring is optimally J-clean. In view of [10, Theorem 20], uniquely clean rings are strongly J-clean. Further, they have the property that all idempotents are central. Hence, the additional condition for optimally clean is automatically satisfied (just take $c = a$). Strongly clean power-series over noncommutative rings have ever been studied by Shifflet [12]. We now establish strong J-cleanness of power series over a general ring.

Lemma 4.1 *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

- (1) $a \in R$ is *optimally J-clean*.
- (2) *There exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in eR(1 - e) + (1 - e)Re$ such that $[a, c] + [e, b] = 0$.*

Proof (1) \Rightarrow (2) Since $a \in R$ is optimally J-clean, there exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in R$ such that

$[a, c] = [e, b]$. It is easy to check that

$$\begin{aligned}
[a, ec(1-e) + (1-e)ce] &= [a, ec(1-e)] + [a, (1-e)ce] \\
&= e[a, c](1-e) + (1-e)[a, c]e \\
&= e[e, b](1-e) + (1-e)[e, b]e \\
&= [e, b],
\end{aligned}$$

and therefore

$$[a, -ec(1-e) - (1-e)ce] + [e, b] = 0.$$

(2) \Rightarrow (1) There exists an idempotent $e \in R$ such that $a - e \in J(R)$ and $ae = ea$, and that for any $b \in R$, there exists $c \in eR(1-e) + (1-e)Re$ such that $[a, c] + [e, b] = 0$. Choose $c' = -c$. Then $[a, c'] = [e, b]$, as required. \square

The following two lemmas are taken from a thesis and not a published paper [12], and so we include simple proofs to indicate how to get these results

Lemma 4.2 [12, Lemma 3.2.1] *Let R be a ring, and let $n \geq 2$. If $e_0 = e_0^2 \in R$ and $e_k(1 - e_0) = \sum_{i=0}^{k-1} e_i e_{k-i}$ ($0 < k < n$), then $e_0 \left(\sum_{i=1}^{n-1} e_i e_{n-i} \right) = \left(\sum_{i=1}^{n-1} e_i e_{n-i} \right) e_0$.*

Proof Straightforward. \square

Lemma 4.3 [12, Theorem 3.2.2] *Let R be a ring, and let $n \geq 2$. If $e_0 = e_0^2 \in R$, $e_k(1 - e_0) = \sum_{i=0}^{k-1} e_i e_{k-i}$ and $[r_0, e_k] + [r_1, e_{k-1}] + \cdots + [r_k, e_0] = 0$ for all $0 < k < n$. Then*

$$\left[r_0, \sum_{i=1}^{n-1} e_i e_{n-i} \right] = (1 - e_0) \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) - \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) e_0.$$

Proof Let $t_n = \sum_{i=1}^{n-1} [e_i, r_{n-i}]$, $\alpha_k = e_1 r_{k-1} + \cdots + e_{k-1} r_1$ and $\beta_k = r_{k-1} e_1 + \cdots + r_1 e_{k-1}$. Then $\alpha_n e_0 + e_0 \alpha_n = \alpha_n - \gamma_n + e_{n-1} r_0$ where $\gamma_n = (e_1 r_{n-2} + e_2 r_{n-3} + \cdots + e_{n-1} r_0) e_1 + (e_1 r_{n-3} + e_2 r_{n-4} + \cdots + e_{n-2} r_0) e_2 + \cdots + (e_1 r_0) e_{n-1}$. Likewise, $\beta_n e_0 + e_0 \beta_n = \beta_n - \lambda_n + r_0 e_{n-1}$, where $\lambda_n = e_1 (r_{n-2} e_1 + r_{n-3} e_2 + \cdots + r_0 e_{n-1}) + e_2 (r_{n-3} e_1 + r_{n-4} e_2 + \cdots + r_0 e_{n-2}) + \cdots + e_{n-1} (r_0 e_1)$. Further, we verify that $\gamma_n = \lambda_n$. Therefore $e_0 t_n + t_n e_0 = t_n + e_{n-1} r_0 - r_0 e_{n-1}$, and so $e_0 \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) + \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) e_0 = \sum_{i=1}^{n-1} [e_i, r_{n-i}] - \left[r_0, \sum_{i=1}^{n-1} e_i e_{n-i} \right]$, as needed. \square

In [12, Theorem 3.2.2], Shifflet characterized strongly clean power series in terms of optimal cleanness. Following a similar route, we now modify Shifflet's method and apply to strongly J-clean power series by means of optimal J-cleanness.

Theorem 4.4 *Let R be a ring, and let $f(x) \in R[[x]]$. If $f(0) \in R$ is optimally J -clean, then $f(x) \in R[[x]]$ is strongly J -clean.*

Proof Write $f(x) = \sum_{i=0}^{\infty} r_i x^i$. Then we can find an idempotent e_0 such that $r_0 = e_0 + (r_0 - e_0)$ is an optimally J -clean decomposition of r_0 . In view of Lemma 4.1, there exists some $e_1 \in (1 - e_0)Re_0 + e_0R(1 - e_0)$ such that $[r_0, e_1] + [e_0, r_1] = 0$. Clearly, $e_1 = e_0e_1 + e_1e_0$. We shall prove that there exist $e_2, \dots, e_k, \dots \in R$ such that

$$e_k = e_0e_k + e_1e_{k-1} + \dots + e_ke_0 \text{ and } [r_0, e_k] + [r_1, e_{k-1}] + \dots + [r_k, e_0] = 0.$$

Assume that this is true for all $1 \leq k \leq n-1$. Set $f_n = (1 - 2e_0)(e_1e_{n-1} + e_2e_{n-2} + \dots + e_{n-1}e_1)$ and $s_n = r_n + [e_0, [e_1, r_{n-1}] + [e_2, r_{n-2}] + \dots + [e_{n-1}, r_1]]$. By virtue of Lemma 4.1, we have some $g_n \in (1 - e_0)Re_0 + e_0R(1 - e_0)$ such that $[r_0, g_n] = [e_0, s_n]$. Let $e_n = f_n + g_n$. In light of Lemma 4.2, analogously to [12, Theorem 3.2.2], we obtain

$$\sum_{i=1}^{n-1} e_ie_{n-i} = (1 - e_0)e_n - e_ne_0.$$

Thus, $e_n = \sum_{i=1}^n e_ie_{n-i}$. Furthermore, that

$$\begin{aligned} [r_0, f_n] &= [r_0, (1 - e_0)(\sum_{i=1}^{n-1} e_ie_{n-i})] - [r_0, (\sum_{i=1}^{n-1} e_ie_{n-i})e_0] \\ &= (1 - e_0)[r_0, (\sum_{i=1}^{n-1} e_ie_{n-i})](1 - e_0) - e_0[r_0, (\sum_{i=1}^{n-1} e_ie_{n-i})]e_0. \end{aligned}$$

By using Lemma 4.3, we have

$$[r_0, \sum_{i=1}^{n-1} e_ie_{n-i}] = (1 - e_0)(\sum_{i=1}^{n-1} [e_i, r_{n-i}]) - (\sum_{i=1}^{n-1} [e_i, r_{n-i}])e_0,$$

and then

$$[r_0, f_n] = (1 - e_0)(\sum_{i=1}^{n-1} [e_i, r_{n-i}])(1 - e_0) + e_0(\sum_{i=1}^{n-1} [e_i, r_{n-i}])e_0.$$

Moreover,

$$\begin{aligned} [r_0, g_n] &= [e_0, s_n] \\ &= [e_0, r_n] + [e_0, [e_0, \sum_{i=1}^{n-1} [e_i, r_{n-i}]]] \\ &= [e_0, r_n] + e_0(\sum_{i=1}^{n-1} [e_i, r_{n-i}])(1 - e_0) + (1 - e_0)(\sum_{i=1}^{n-1} [e_i, r_{n-i}])e_0. \end{aligned}$$

Thus, we get

$$\begin{aligned}
& [r_0, e_n] \\
&= [r_0, f_n] + [r_0, g_n] \\
&= [e_0, r_n] + e_0 \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) (1 - e_0) + (1 - e_0) \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) e_0 \\
&+ (1 - e_0) \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) (1 - e_0) + e_0 \left(\sum_{i=1}^{n-1} [e_i, r_{n-i}] \right) (1 - e_0) \\
&= \sum_{i=0}^{n-1} [e_i, r_{n-i}];
\end{aligned}$$

hence that $\sum_{i=0}^n [r_i, e_{n-i}] = 0$. By induction, the claim is true. Thus, $\sum_{i=0}^{\infty} e_i x^i = \left(\sum_{i=0}^{\infty} e_i x^i \right)^2 \in R[[x]]$ and $f(x) \left(\sum_{i=0}^{\infty} e_i x^i \right) = \left(\sum_{i=0}^{\infty} e_i x^i \right) f(x)$. Since $f(0) - e(0) \in J(R)$, we see that $f(x) - \sum_{i=0}^{\infty} e_i x^i \in J(R[[x]])$. Therefore $f(x) \in R[[x]]$ is strongly J-clean, as asserted. \square

Corollary 4.5 *Let R be an abelian ring, and let $f(x) \in R[[x]]$. If $f(0) \in R$ is strongly J-clean, then $f(x) \in R[[x]]$ is strongly J-clean.*

Proof Suppose $f(0) \in R$ is strongly J-clean. Then there exists an idempotent $e \in R$ such that $f(0) - e \in J(R)$ and $f(0)e = ef(0)$. For any $b \in R$, we choose $c = 0 \in eR(1 - e) + (1 - e)Re$. Then $[f(0), c] + [e, b] = 0$; hence that $f(0)$ is J-optimally clean. Therefore $f(x) \in R[[x]]$ is strongly J-clean, in terms of Theorem 4.4. \square

Corollary 4.6 *Let R be a ring, and let $f(x) \in R[[x]]$. Then the following are equivalent:*

- (1) $f(0) \in R$ is optimally J-clean.
- (2) $f(x) \in R[[x]]$ is optimally J-clean.

Proof (1) \Rightarrow (2) In view of Theorem 4.4, $f(x) \in R[[x]]$ is strongly J-clean. Hence, there exists an idempotent $e(x) \in R[[x]]$ such that $w(x) := f(x) - e(x) \in J(R[[x]])$ and $f(x)e(x) = e(x)f(x)$. Thus, $f(x) = (1 - e(x)) + (2e(x) - 1 + w(x))$. As $(2e(x) - 1)^2 = 1$, we see that $(2e(x) - 1 + w(x)) = (2e(x) - 1)(1 + (2e(x) - 1)w(x)) \in U(R[[x]])$. By virtue of [12, Theorem 3.3.2], $f(x) \in R[[x]]$ is optimally clean. For any $b(x) \in R[[x]]$, there exists $c(x) \in R[[x]]$ such that $[f(x), -c(x)] = [1 - e(x), b(x)]$. This implies that $[f(x), c(x)] = [e(x), b(x)]$. Therefore $f(x) \in R[[x]]$ is optimally J-clean, as desired.

(2) \Rightarrow (1) This is obvious. \square

Lemma 4.7 *Let R be a ring, and let $u \in U(R)$. If $uau^{-1} \in R$ is J-optimally clean, then so is a in R .*

Proof Since $uau^{-1} \in R$ is J-optimally clean, there exists an idempotent $e \in R$ such that $uau^{-1} - e \in J(R)$ and $(uau^{-1})e = e(uau^{-1})$, and that for any $b \in R$, there exists a $c \in R$ such that $[uau^{-1}, c] = [e, u^{-1}bu]$. Thus, $a - u^{-1}eu \in J(R)$ and $a(u^{-1}eu) = (u^{-1}eu)a$. Furthermore, $[a, uau^{-1}] = [u^{-1}eu, b]$. Accordingly, $a \in R$ is J-optimally clean. \square

Lemma 4.8 *Let R be a 2-projective-free wb-ring, and let $A \in M_2(R)$. Then the following are equivalent:*

- (1) $A \in M_2(R)$ is optimally J-clean.
- (2) $A \in M_2(R)$ is strongly J-clean.

Proof (1) \Rightarrow (2) This is obvious.

(2) \Rightarrow (1) Suppose that $A \in M_2(R)$ is strongly J-clean. In view of Theorem 3.1, $A \in J(M_2(R))$, or $I_2 - A \in J(M_2(R))$, or A is similar to a diagonal matrix $\text{diag}(\alpha, \beta)$ with $\alpha \in J(R), \beta \in 1 + J(R)$. If $A \in J(M_2(R))$, then $A - 0 = A \in J(M_2(R))$. For any $B \in M_2(R)$, we have $[A, 0] = [0, B]$, and so $A \in M_2(R)$ is optimally J-clean. If $I_2 - A \in J(M_2(R))$, then $A - I_2 \in J(M_2(R))$. For any $B \in M_2(R)$, we see that $[A, 0] = [I_2, B]$, and then $A \in M_2(R)$ is optimally J-clean. Hence, we have a $P \in GL_2(R)$ such that $PAP^{-1} = \text{diag}(\alpha, \beta)$, where $\alpha \in J(R)$ and $\beta \in 1 + J(R)$. For any $B = (b_{ij}) \in M_2(R)$, by hypothesis, we have $c_1, c_2 \in R$ such that

$$\alpha c_1 - c_1 \beta = -b_{12} \text{ and } \beta c_2 - c_2 \alpha = b_{21}.$$

Set $C = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}$. Then

$$[\text{diag}(\alpha, \beta), C] = \begin{pmatrix} 0 & -b_{12} \\ b_{21} & 0 \end{pmatrix} = [\text{diag}(0, 1), B].$$

One easily checks that $\text{diag}(\alpha, \beta) - \text{diag}(0, 1) \in J(M_2(R))$. Therefore PAP^{-1} is optimally J-clean. In light of Lemma 4.7, $A \in M_2(R)$ is J-optimally clean. \square

Strongly J-clean matrices over local rings were studied in [4, Theorem 3.2], but the proof there depends on the local property. We now have at our disposal all the information necessary to prove the following.

Theorem 4.9 *Let R be a 2-projective-free wb-ring. Then the following are equivalent:*

(1) $A(x) \in M_2(R[[x]])$ is strongly J -clean.

(2) $A(0) \in M_2(R)$ is strongly J -clean.

Proof (1) \Rightarrow (2) By hypothesis, there exists an idempotent $E(x) \in M_2(R[[x]])$ such that $A(x) - E(x) \in J(M_2(R[[x]]))$ and $E(x)A(x) = A(x)E(x)$. This implies that $A(0) - E(0) \in J(M_2(R))$ and $A(0)E(0) = E(0)A(0)$. Therefore, proving (2).

(2) \Rightarrow (1) Since $A(0) \in M_2(R)$ is strongly J -clean, by virtue of Lemma 4.8, $A(0) \in M_2(R)$ is J -optimally clean. According to Theorem 4.4, $A(x) \in M_2(R[[x]])$ is strongly J -clean. \square

Corollary 4.10 *Let R be a 2-projective-free ring. If $J(R)$ is nil, then the following are equivalent:*

(1) $A(x) \in M_2(R[[x]])$ is strongly J -clean.

(2) $A(0) \in M_2(R)$ is strongly J -clean.

Proof Let $\alpha \in 1 + J(R), \beta \in J(R)$. Write $\beta^n = 0$. Choose $\varphi = \ell_{\alpha^{-1}} + \ell_{\alpha^{-2}r\beta} + \cdots + \ell_{\alpha^{-n}r\beta^{n-1}} : R \rightarrow R$. For any $r \in R$, one easily checks that

$$\begin{aligned} & (\ell_{\alpha} - r_{\beta})\varphi(r) \\ &= (\ell_{\alpha} - r_{\beta})(\alpha^{-1}r + \alpha^{-2}r\beta + \cdots + \alpha^{-n}r\beta^{n-1}) \\ &= (r + \alpha^{-1}r\beta + \cdots + \alpha^{-n+1}r\beta^{n-1}) - (\alpha^{-1}r\beta + \alpha^{-2}r\beta^2 + \cdots + \alpha^{-n+1}r\beta^{n-1}) \\ &= r. \end{aligned}$$

Thus, $(\ell_{\alpha} - r_{\beta})\varphi = 1_R$. Thus, $\ell_{\alpha} - r_{\beta} : R \rightarrow R$ is surjective. Likewise, $\ell_{\beta} - r_{\alpha} : R \rightarrow R$ is surjective. Hence, R is a wb-ring. This completes the proof, by Theorem 4.9. \square

Corollary 4.11 *Let R be a PSF ring. Then the following are equivalent:*

(1) $A(x) \in M_2(R[[x]])$ is strongly J -clean.

(2) $A(0) \in M_2(R)$ is strongly J -clean.

Proof Since R is PSF, it follows by Theorem 2.5 that R is 2-projective-free. On the other hand, R is a wb-ring, as it is commutative. According to Theorem 4.9, we establish the result. \square

Acknowledgements

The authors are grateful to the referee for his/her useful suggestions which correct several errors and improve many statements, and make the new version more clearer.

References

- [1] D.D. Anderson and V.P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, *Comm. Algebra*, **30**(2002), 3327–3336.
- [2] N. Ashrafi and E. Nasibi, Strongly J-clean group rings, *Proc. Romanian Acad., Series A*, **14**(2013), 9–12.
- [3] W.D. Burgess, On strongly clean matrices over commutative clean rings, arXiv: 1401.2052v1 [math.R.A] 9 Jul 2014.
- [4] H. Chen, *Rings Related Stable Range Conditions*, Series in Algebra 11, World Scientific, Hackensack, NJ, 2011.
- [5] H. Chen, Strongly J-clean matrices over local rings, *Comm. Algebra*, **40**(2012), 1352–1362.
- [6] P.M. Cohn, *Free Rings and their Relations*, Second edition. London Mathematical Society Monographs 19, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1985.
- [7] J. Cui and J. Chen, Pseudopolar matrix rings over local rings, *J. Algebra Appl.*, **13**(2014), 1350109 [12 pages] DOI: 10.1142/S0219498813501090.
- [8] A.J. Diesl and T.J. Dorsey, Strongly clean matrices over arbitrary rings, *J. Algebra*, **399**(2014), 854–869.
- [9] B. Li, Strongly clean matrix rings over noncommutative local rings, *Bull. Korean Math. Soc.*, **46**(2009), 71–78.
- [10] W.K. Nicholson and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit, *Glasgow Math. J.*, **46**(2004), 227–236.
- [11] J. Rosenberg, *Algebraic K-Theory and Its Applications*, Graduate Texts in Mathematics, **147**, Springer-Verlag, New York, 1994.
- [12] D.R. Shifflet, Optimally Clean Rings, Ph.D. Thesis, Bowling Green State University, Bowling Green, 2011.
- [13] X. Yang and Y. Zhou, Strong cleanness of the 2×2 matrix ring over a general local ring, *J. Algebra*, **320**(2008), 2280–2290.

Marjan Sheibani Abdolyousefi
Faculty of Mathematics, Statistics and Computer Science
Semnan University, Semnan, Iran
Email: m.sheibani1@gmail.com

Corresponding author.
Huanyin Chen
Department of Mathematics
Hangzhou Normal University
Hangzhou, 310036, China
Email: huanyinchen@aliyun.com

Rahman Bahmani Sangesari
Faculty of Mathematics, Statistics and Computer Science
Semnan University, Semnan, Iran
rbahmani@semnan.ac.ir